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Representations of the Weyl group in spaces of square integrable functions with respect to *p*-adic valued Gaussian distributions

Sergio Albeverio and Andrew Khrennikov
Mathematical Institute, Ruhr-University, D-44780 Bochum, Germany†

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Abstract. We construct a representation of the Weyl group in the p-adic Hilbert space of functions which are square integrable with respect to a p-adic valued Gaussian distribution. The operators corresponding to position and momentum are determined by groups of unitary operators with parameters restricted to some balls in the field Q_p of p-adic numbers. A surprising fact is that the radii of these balls are connected by 'an uncertainty relation' which can be considered as a p-adic analogue of the Heisenberg uncertainty relations. The p-adic Hilbert space representation of the Weyl group is the basis for a calculus of pseudo-differential operators and for an operator quantization over p-adic numbers.

1. Introduction

Interest in the physics of non-Archimedean quantum models [1–13] is based on the idea that the structure of space–time for very short distances (less than 'Planck's length') might conveniently be described in terms of non-Archimedean numbers. There are different mathematical ways to describe this violation of the Archimedean axiom. One is given by non-standard analysis [2, 14, 15], where a non-Archimedean extension ${}^{\star}R$ of the field of real numbers R is used. See [1–4] for applications of methods of non-standard analysis to mathematical physics. Another one is p-adic analysis [16–21], where complete non-Archimedean p-adic extensions Q_p of the incomplete field of rational numbers Q are used. See [5–13, 22, 23] for applications of p-adic numbers in mathematical physics.

Representations of groups in Hilbert spaces is one of the cornerstones of ordinary quantum mechanics. It is very natural to try to develop p-adic quantum mechanics in a similar way. In the present paper we construct a representation of the Weyl–Heisenberg group in a p-adic Hilbert space (see [6, 24, 25] for p-adic Hilbert spaces), the space $L_2(Q_p, \nu_b)$ of L_2-functions with respect to a p-adic valued Gaussian distribution ν_b (the symbol b indicates the covariance function). Here the situation differs very much from the one of ordinary quantum mechanics. If we denote by $U(\alpha)$ and $V(\beta)$ the groups of unitary operators corresponding to position and momentum operators respectively, then these groups are defined only for parameters α and β belonging to balls $\mathcal{U}_{R(b)}$ and $\mathcal{U}_{r(b)}$, respectively, where $R(b)$ and $r(b)$ depend on the covariance b of the Gaussian distribution. Moreover these radii are connected by a p-adic analogue of the Heisenberg uncertainty relation.

We also study the representation of the translation group in the space $L_2(Q_p, \nu_b)$. Here the result differs also from the one of ordinary quantum mechanics and it is more similar

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to the one which holds in quantum field theory where Gaussian distributions on infinite dimensional spaces are used. If μ is a Gaussian measure on an infinite dimensional Hilbert space $\mathcal{H}$ then we cannot construct a representation of translations from all of $\mathcal{H}$ in $L_2(\mathcal{H}, \mu)$ because of the well known fact that the translation μ^h of a Gaussian measure on $\mathcal{H}$ by a vector $h \in H$ may be singular with respect to μ, see, for example, [26] (and [2, 27] in connection with problems of quantum fields). It is well known that μ^h is equivalent to μ if and only if h belongs to a certain proper ('Cameron–Martin') subspace. In a similar way we cannot construct in the space $L_2(Q_p, \nu_b)$ a representation of translations by all elements h in Q_p, in fact we have to restrict the considerations to translations belonging to some ball (which is an additive subgroup in Q_p), whose radius depends on b. This fact is connected with the non-existence of translation invariant measures in the p-adic case (similarly as for infinite-dimensional spaces over the field of real numbers).

2. Groups of unitary isometric operators in a *p*-adic Hilbert space

The field of real numbers R is constructed as the completion of the field of rational numbers Q with respect to the metric $\rho(x, y) = |x - y|$, where $|\cdot|$ is the usual valuation given by the absolute value. The fields of p-adic numbers Q_p are constructed in a corresponding way, by using other valuations. For any prime number the p-adic valuation $|\cdot|_p$ is defined in the following way. At the first, we define it for natural numbers. Every natural number n can be represented as the product of prime numbers: $n = 2^{r_2}3^{r_3}\cdots p^{r_p}\cdots$. Then we define $|n|_p = p^{-r_p}$, we set $|0|_p = 0$ and $|-n|_p = |n|_p$. We extend the definition of p-adic valuation $|\cdot|_p$ to all rational numbers by setting for $m \neq 0 : |n/m|_p = |n|_p/|m|_p$. The completion of Q with respect to the metric $\rho_p(x, y) = |x - y|_p$ is a locally compact field Q_p.

Two valuations $|\cdot|_\alpha$ and $|\cdot|_\beta$ on Q are said to be equivalent if there exists a positive real number c such that $|\cdot|_\alpha = |\cdot|_\beta^c$. It is well known, see [16–21], that $|\cdot|$ and $|\cdot|_p$ are the only possible non-trivial valuations on Q up to equivalence.

The p-adic valuation satisfies the strong triangle inequality: $|x + y|_p \leqslant \max(|x|_p, |y|_p)$. Set $\mathcal{U}_r(a) = \{x \in Q_p : |x - a|_p \leqslant r\}, a \in Q_p, r = p^n, n = 0, \pm 1, \pm 2, \ldots$. This is by definition the ball in Q_p with the centre in a of radius r. Balls are in the same time closed and open. Set $\mathcal{U}_r \equiv \mathcal{U}_r(0)$.

For any $x \in Q_p$ we have a unique canonical expansion (converging in the $|\cdot|_p$-norm) of the form

$$x = a_{-n}/p^n + \cdots\ a_0 + \cdots + a_k p^k + \cdots \tag{1}$$

where $a_j = 0, 1, \ldots, p - 1$, are the 'digits' of the p-adic expansion.

The following elementary result of p-adic analysis will be useful below: a series $\sum w_n, w_n \in Q_p$, converges *iff* $|w_n|_p \to 0, n \to \infty$.

Using the definition of the p-adic valuation, we get $|n|_p \leqslant 1$ for every natural number n. Thus, the sequence $|n!|_p$ is decreasing. Moreover we have [16–21]:

$$p^{(n-)/(1-p)} \leqslant |n!|_p \leqslant npp^{n/(1-p)}\,. \tag{2}$$

Using this estimate, we get that the exponential series

$$\mathrm{e}^x = \sum_{n=0}^{\infty} x^n/n!$$

converges iff $|x|_p < p^{1/(1-p)}$. If $p \neq 2$, then we can rewrite this condition as $|x|_p < 1$ or $x \in \mathcal{U}_{1/p}(0)$. If $p = 2$, then we have $|x|_2 < \frac{1}{2}$ or $x \in \mathcal{U}_{1/4}(0)$.

Let $p = 3, \bmod 4$, then the equation $x^2+1 = 0$ has no solutions in Q_p. We can consider the quadratic extension $Q_p(\mathrm{i})$ with $\mathrm{i} = \sqrt{-1}$ of Q_p. In analogy with complex numbers we set $z = x + \mathrm{i}y, x, y \in Q_p, \mathrm{i} = \sqrt{-1}$, and $\bar{z} = x - \mathrm{i}y$. The valuation on $Q_p(\mathrm{i})$ is also denoted by $|\cdot|_p$, $|z|_p = \sqrt{||z|^2|_p}$, where $|z|^2 = z\bar{z} = x^2 + y^2$. We remark that $|z|^2$ assumes its values in the field Q_p, whereas $|z|_p$ assumes its values in the field of real numbers.

The definition of p-adic Hilbert spaces [6, 28, 29] is based on a coordinate representation (an analogue of l_2). For the sequence $\lambda = (\lambda_n) \in Q_p^\infty$, $\lambda_n \neq 0$ we set

$$\mathcal{H}_\lambda = \left\{ f = (f_n) \in Q_p^\infty : \text{ the series } \sum f_n^2 \lambda_n \text{ converges in } Q_p \right\} .$$

We have

$$\mathcal{H}_\lambda = \left\{ f = (f_n) \in Q_p^\infty : \lim_{n\to\infty} |f_n|_p \sqrt{|\lambda_n|_p} = 0 \right\} .$$

In the space $\mathcal{H}_\lambda$ we introduce the norm $||f||_\lambda = \max_n |f_n|_p \sqrt{|\lambda_n|_p}$. The space $\mathcal{H}_\lambda$ is a non-Archimedean Banach space in the sense of [19] (over the field Q_p). On the space $\mathcal{H}_\lambda$ we introduce a 'p-adic valued inner product ' $(\cdot,\cdot)$ consistent with a ' p-adic length ' $|f|_\lambda^2 = \sum f_n^2 \lambda_n$ by setting $(f, g)_\lambda = \sum f_n g_n \lambda_n$. The p-adic inner product $(\cdot,\cdot)_\lambda : \mathcal{H}_\lambda \times \mathcal{H}_\lambda \to Q_p$ is continuous and the *Cauchy–Buniakovski–Schwarz inequality* holds

$$|(f, g)_\lambda|_p \leqslant ||f||\,||g|| . \tag{3}$$

(see [6]).

Definition 1. The triplet $(\mathcal{H}_\lambda, (\cdot,\cdot)_\lambda, ||\cdot||_\lambda)$ is called a p-adic coordinate Hilbert space.

More generally we shall define a p-adic inner product on a Q_p-linear space E as an arbitrary non-degenerate symmetric bilinear form $(\cdot,\cdot) : E \times E \to Q_p$.

Remark. It is impossible to introduce a p-adic analogue of the positive definiteness of a bilinear form. For instance, for the field of p-adic numbers any element $\gamma \in Q_p$ can be represented as $\gamma = (x, x)_\lambda$, $x \in \mathcal{H}_\lambda$ (this is a simple consequence of properties of bilinear forms over Q_p, [18]).

The triplets $(E_j, (\cdot,\cdot)_j, ||\cdot||_j)$, $j = 1, 2$, where E_j are non-Archimedean Banach spaces, $||\cdot||_j$ are norms and $(\cdot,\cdot)_j$ are inner products satisfying (3), are said to be isomorphic if the spaces E_1 and E_2 are algebraically isomorphic and the algebraic isomorphism $I : E_1 \to E_2$ is isometric and unitary, i.e. $||Ix||_2 = ||x||_1$, $(Ix, Iy)_2 = (x, y)_1$.

Definition 2. The triplet $(E, (\cdot,\cdot), ||\cdot||)$ is a p-adic Hilbert space if it is isomorphic to the coordinate Hilbert space $(\mathcal{H}_\lambda, (\cdot,\cdot)_\lambda, ||\cdot||_\lambda)$ for a certain λ.

The isomorphic relation divides the family of all p-adic Hilbert spaces into equivalence classes. An equivalence class is characterized by some coordinate representative $\mathcal{H}_\lambda$. It is an unsolved mathematical problem to classify all p-adic Hilbert spaces.

Hilbert spaces over the quadratic extensions $Q_p(\mathrm{i})$ of the Q_p can be introduced in an analogous way. For a given sequence $\lambda = (\lambda_n) \in Q_p^\infty$, $\lambda_n \neq 0$, we set

$$\mathcal{H}_\lambda = \{ f = (f_n) \in Q_p(\mathrm{i})^\infty : \text{ the series} \sum |f_n|^2 \lambda_n \text{ converges in the field } Q_p \}$$

$$= \{ f = (f_n) : \lim_{n\to\infty} |f_n|_p \sqrt{|\lambda_n|_p} = 0 \}$$

$$||f||_\lambda = \max_n |f_n|_p \sqrt{|\lambda_n|_p}$$

$$(f, g)_\lambda = \sum f_n \bar{g}_n \lambda_n \qquad |f|_\lambda^2 = (f, f)_\lambda = \sum |f_n|^2 \lambda_n \in Q_p .$$

The triplet $(\mathcal{H}_\lambda,\ (\cdot,\cdot)_\lambda,\ ||\cdot||_\lambda)$ is a p-adic complex coordinate Hilbert space. A general p-adic complex Hilbert space $(E,\ (\cdot,\cdot),\ ||\cdot||)$ is by definition isomorphic to some p-adic complex coordinate Hilbert space.

Remark. We can generalize all results of this paper to consider an analogue of a Hilbert space using an arbitrary complete field K with non-trivial non-Archimedean valuation $|\cdot|_K$ instead of the field of p-adic numbers Q_p (and one of quadratic extensions $K(\sqrt{\tau})$ instead of $Q_p(\mathrm{i})$). We wish to note that a valuation $|\cdot|_K$ is said to be a non-Archimedean one if satisfies to the strong triangle inequality: $|x+y|_K \leqslant \max[|x|_K, |y|_K]$. We wish also to note that a non-Archimedean number field may have a number of non-isomorphic quadratic extensions. In particular, Q_p, $p \neq 2$, has three non-isomorphic quadratic extensions and Q_2 has seven non-isomorphic quadratic extensions. Therefore, also in the p-adic case we may study representations of the Weyl group, not only in the p-adic Hilbert space over $Q_p(\mathrm{i})$, but also in p-adic Hilbert spaces over other quadratic extensions. These representations are not equivalent. The speculations on possible physical interpretations of the non-equivalent representations corresponding to different quadratic extensions were presented in [6, 28, 29].

The mathematical theory of p-adic Hilbert spaces is only in its beginnings, most attention having been given up to now to p-adic Banach spaces [17, 19–21]. To develop a physical formalism similar to the one used in ordinary quantum mechanics, it is useful to have the additional structure of a Hilbert space (see [22, 23] for a probabilistic interpretation of the p-adic inner product).

The first non-Archimedean analogue of a Hilbert space was considered by Kalisch [24]. But a class of non-Archimedean Hilbert spaces introduced in [24] is too restrictive for our applications. Kalisch introduced Hilbert spaces over a complete separable non-Archimedean field K with the valuation $|\cdot|_K$ which satisfies the following conditions: (K1) $|2|_K = 1$; (K2) every $x \in K$, $|x|_K = 1$, (a unit of K) possesses a square root in K. The last condition is very strong. In particular, Q_p and $Q_p(\mathrm{i})$ do not satisfy this condition. The only interesting example of a non-Archimedean field which satisfies the condition (K2) is the completion C_p of the algebraic closure Q_p^a of Q_p. But this field is not useful for our applications since it is an infinite dimensional space over Q_p and there is no continuous involutions on C_p.

Remark. We may try to extend our formalism and use elements of the Galious group $G(C_p/Q_p)$ instead of an involution. But this theory is much more complicated.

Now let K be a non-Archimedean field which satisfies the above restrictions. Kalisch defined a non-Archimedean Hilbert space as a triple $(E, (\cdot,\cdot),\ \|\cdot\|)$ where $(E, \|\cdot\|)$ is a separable non-Archimedean Banach space over K, $(\cdot,\cdot) : E \times E \to K$ is a symmetric bilinear form which satisfies to the following conditions: (K3) the Cauchy–Buniakovski–Schwarz inequality (3) holds; (K4) for every $x \in E$ there exists $\alpha \in K$ such that $\|x\| = |\alpha|_K$; (K5) for every $x \in E$ there exists $x', x' \neq 0$, such that $|(x, x')|_K = \|x\|\|x'\|$.

Kalisch proved that every non-Archimedean Hilbert space is isomorphic to the coordinate Hilbert space over K:

$$\mathcal{H}(K) = \{f = (f_n) \in K^\infty : \lim_{n\to\infty} f_n = 0\}.$$

We wish to note that our p-adic (and complex p-adic) Hilbert spaces do not satisfy to the condition (K4). An extended review on different non-Archimedean analogues of a Hilbert space is contained in [25]. We wish to note that our class of p-adic Hilbert spaces does not coincide with anyone considered in [25].

Denote the space of bounded operators $A : \mathcal{H} \to \mathcal{H}$ in a p-adic Hilbert space $\mathcal{H}$ by the symbol $\mathcal{L}(\mathcal{H})$ with norm $\|A\| = \sup_{x\neq 0} \|Ax\|/\|x\|$. $\mathcal{L}(\mathcal{H})$ is a non-Archimedean Banach space, $\|A+B\| \leqslant \max[\|A\|, \|B\|]$ and $\|A+B\| = \max[\|A\|, \|B\|]$, if $\|A\| \neq \|B\|$. Denote the group of linear isometries of the p-adic Hilbert space $\mathcal{H}$ by the symbol $IS(\mathcal{H}) : \|Ax\| = \|x\|$ and $A(\mathcal{H}) = \mathcal{H}$. An operator $U \in \mathcal{L}(\mathcal{H})$ is said to be a unitary operator, if $(Ux, Uy) = (x, y)$ for all $x, y \in \mathcal{H}$ and $U(\mathcal{H}) = \mathcal{H}$. Denote the group of unitary operators in $\mathcal{H}$ by the symbol $UN(\mathcal{H})$.

Remark. At the moment, we do not know whether every unitary operator in a p-adic Hilbert space is isometric.

Set $UI(\mathcal{H}) = IS(\mathcal{H}) \cap UN(\mathcal{H})$. A bounded operator A in a p-adic Hilbert space $\mathcal{H}$ is said to be symmetric iff $(Ax, y) = (x, Ay)$ for all $x, y \in \mathcal{H}$.

Remark. A p-adic Hilbert space $\mathcal{H}$ is not isomorphic to its dual space $\mathcal{H}'$. Hence if A is an operator in $\mathcal{H}$ its adjoint operator $A^\star$ acts in $\mathcal{H}'$ and it is not clear what is the analogue of a self-adjoint operator. On the other hand, as we shall see below some basic operators of p-adic quantum mechanics are bounded and symmetric in the above sense. For this reason we shall restrict our considerations to bounded symmetric operators. We also remark that if x is an eigenvector of the symmetric bounded operator A, $Ax = \lambda x$, and $(x, x) \neq 0$ then λ belongs to Q_p (this is a situation very similar to the case of complex Hilbert spaces with an indefinite metric). It seems that the spectral theory of symmetric operators in p-adic Hilbert spaces is not yet well developed.

Note that every ball $\mathcal{U}_r$ in Q_p is an additive subgroup of Q_p. A map $F : \mathcal{U}_r \to \mathcal{L}(\mathcal{H})$ with the properties $F(\alpha+\beta) = F(\alpha)F(\beta), F(0) = I, \alpha, \beta \in \mathcal{U}_r$, where I is the unit operator in $\mathcal{H}$ is said to be a one parameter group of operators. If we consider $IS(\mathcal{H}), UN(\mathcal{H}), UI(\mathcal{H})$ instead of $\mathcal{L}(\mathcal{H})$ we get the definitions of parametric groups of isometric, unitary and isometric unitary operators respectively. If the map $F : \mathcal{U}_r \to \mathcal{L}(\mathcal{H})$ is analytic, the one parameter group is called analytic.

Let a belong to R_+. Set $[a]_p^- = \sup\{\lambda = p^k, k = 0, \pm 1, \ldots : \lambda < a\}$ and $\gamma(A) = 1/\|A\| p^{1/(p-1)}$. The following proposition is a consequence of the estimate (2).

Proposition 2.1. Let an operator A belongs to $\mathcal{L}(\mathcal{H})$. The map $\alpha \to \mathrm{e}^{\alpha A}, \alpha \in \mathcal{U}_r, r = [\gamma(A)]_p^-$, is an analytic one parameter group of isometric operators.

Proof. As $\mathrm{e}^{\alpha A} = I + \sum_{m=1}^{\infty} (\alpha A)^n/n! = I + \sum_{m=1}^{\infty} A_m$ and $\|A_m\| < 1$ for every $m = 1, 2, \ldots$, we have $\|\mathrm{e}^{\alpha A} x\| = \max(\|x\|, \|A_m x\|) = \|x\|$ (where we have used that $\|A_m x\| < \|x\|$.) Of course, the operator $\mathrm{e}^{\alpha A}$ has an inverse operator, namely $\mathrm{e}^{-\alpha A}$.

Proposition 2.2. Let A be a bounded symmetric operator in $\mathcal{H}$. The map $\alpha \to \mathrm{e}^{i\alpha A}, \alpha \in \mathcal{U}_r, r = [\gamma(A)]_p^-$, is an analytic one parameter group of isometric unitary operators in $\mathcal{H}$.

Proof. The isometric property is proved in the same way as in the previous proposition. To prove unitarity, we need only to perform algebraic computations which do not depend on the number field but only on the fact that $\mathrm{i}^2 = -1$.

3. *p*-adic valued Gaussian integration and spaces of square integrable functions

In [6, 28, 29] a general definition of the p-adic valued Gaussian integral was proposed on the basis of the theory of distributions (a Gaussian distribution was defined as a distribution whose Laplace transform is of the form $\exp\{b(x, x)/2\}$ with $b(x, x)$ a quadratic function). For our applications we can use a simpler approach based on the moments of the Gaussian distribution.

Let b be a p-adic number, $b \neq 0$, the p-adic Gaussian distribution ν_b is defined as a Q_p- linear functional (on the space of polynomials) by its moments

$$M_{2n} = \int_{Q_p} x^{2n} \nu_b(\mathrm{d}x) = (2n)!b^n/n!2^n \qquad n \in \mathbb{N}_0$$

with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$;

$$M_{2n+1} = \int_{Q_p} x^{2n+1} \nu_b(\mathrm{d}x) = 0 \qquad n \in \mathbb{N}_0.$$

In [6] it was shown that these requirements specify ν_b uniquely. We can extend the integration with respect to ν_b to the family $\mathcal{A}$ of all entire analytic functions from Q_p to Q_p. Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n \in Q_p$, be an entire analytic function, i.e. such that the series converges on the whole Q_p. Then by definition

$$\int_{Q_p} g(x) \nu_b(\mathrm{d}x) = \sum_{n=0}^{\infty} a_n M_n \,.$$

It is easy to see that the integral is well defined.

If $|b|_p = p^{2k+1}$ we set $s(b) = p^k$; if $|b|_p = p^{2k}$, we set $s(b) = p^{k-1}$.

Now let us introduce the analogue of Hermite polynomials on Q_p. We define a Hermite polynomial $H_{n,b}(x)$ as the unique polynomial which coincides with the function

$$(-1)^n \mathrm{e}^{x^2/2b} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2/2b} \tag{4}$$

on the ball $\mathcal{U}_{s(b)}$.

In the space of $Q_p(\mathrm{i})$-linear combinations of these polynomials we introduce the inner product

$$(f, g) = \int_{Q_p} f(x) \bar{g}(x) \nu_b(\mathrm{d}x) \,. \tag{5}$$

We then see that $H_{n,b}$ is orthogonal to $H_{n',b}$ for $n \neq n'$ with respect to this inner product and $\int_{Q_p} H_{n,b}^2(x) \nu_b(\mathrm{d}x) = n!/b^n$.

As was done in [6], where the special case $b = \frac{1}{2}$ was considered, it is possible to prove that the Hermite polynomials $\{H_{n,b}(x)\}$ form a basis in the space $\mathcal{A}$ of all entire analytical functions and to introduce the ($Q_p(\mathrm{i})$-valued) space $L_2(Q_p, \nu_b)$ as the completion of $\mathcal{A}$ in the norm given by the above inner product. In fact we have (and this can be taken as definition of $L_2(Q_p, \nu_b)$):

$$L_2(Q_p, \nu_b)$$

$$= \left\{ f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x),\ f_n \in Q_p(\mathrm{i}) :\ \text{the series } \sum_{n=0}^{\infty} |f_n|^2 n!/b^n \text{ converges in } Q_p \right\} .$$

Here we denote the norm in $L_2(Q_p, \nu_b)$ by $\| \cdot \|$.

$L_2(Q_p, \nu_b)$ is a p-adic complex Hilbert space, isomorphic to the complex coordinate Hilbert space $\mathcal{H}_\lambda$ for the weight sequence $\lambda = \{n!/b^n\}$. There is no problem to present examples of b, for which the spaces $L_2(Q_p, \nu_b)$ are not isomorphic, but, at the moment, we cannot solve the general problem of classification of L_2-spaces in the p-adic case.

4. A representation of the translation group

Set $T_\beta(f)(x) = f(x+\beta), \beta \in Q_p$. We shall prove that these operators are bounded for $\beta \in \mathcal{U}_{s(b)}$. Moreover these operators are isometries of $L_2(Q_p, \nu_b)$. Using this we shall construct a representation of the translation group in the p-adic Hilbert space $L_2(Q_p, \nu_b)$.

Lemma 4.1. The formula

$$T_\beta H_{n,b}(x) = \sum_{j=0}^{n} C_n^j (\beta/b)^j H_{n-j,b}(x) \tag{6}$$

holds for the translations of Hermite polinomials.

Proof. This follows immediately from (4). Here $C_n^j = n!/(j!(n-j)!)$ are binomial coefficients.

Theorem 4.1. The operator T_β belongs to $IS(L_2(Q_p, \nu_b))$ for every $\beta \in \mathcal{U}_{s(b)}$ and the map $T : \mathcal{U}_{s(b)} \to IS(L_2(Q_p, \nu_b)), \beta \to T_\beta$, is analytic.

Proof. Using equation (6), we get

$$\begin{aligned} T_\beta(f)(x) &= \sum_{n=0}^{\infty} f_n \sum_{l=0}^{n} C_n^l (\beta/b)^l H_{n-l,b}(x) \\ &= \sum_{l=0}^{\infty} (\beta/b)^l \sum_{n=l}^{\infty} f_n C_n^l H_{n-l,b}(x) \\ &= [I + \sum_{m=1}^{\infty} \beta^m K_m](f)(x) \end{aligned}$$

where I is the unit operator and the operators K_m are defined by

$$K_m(f) = \frac{1}{b^m} \sum_{l=0}^{\infty} C_{m+l}^m f_{m+l} H_{l,b}.$$

We shall now prove that these operators are bounded and get an estimate of their norms. We have

$$\begin{aligned} \|K_m f\|^2 &= (1/|b|_p^{2m}) \max_{0 \leqslant l < \infty} |C_{m+l}^m|_p^2 |f_{m+l}|_p^2 |l!/b^l|_p \\ &= (1/|b|_p^m) \max_{0 \leqslant l < \infty} [|(m+l)!/b^{m+l}|_p |f_{m+l}|_p^2] |(m+l)!/l! m!^2|_p \\ &\leqslant \|f\|^2 (1/|b^m m!|_p) \max_{0 \leqslant l < \infty} |(m+l)!/m!|_p \leqslant (1/|b^m m!|_p) \|f\|^2 . \end{aligned}$$

Thus we have got

$$\|K_m\| \leqslant 1/\sqrt{|b^m m!|_p}$$

and, in particular, $K_m \in \mathcal{L}(L_2(Q_p, \nu_b))$. If $\beta \in \mathcal{U}_{s(b)}$ then

$$|\beta|^m \|K_m\| \leqslant (s(b) p^{1/2(p-1)}/\sqrt{|b|_p})^m = \lambda^m.$$

If $|b|_p = p^{2k+1}$, then $\lambda = p^{1/2(p-1)}/p^{1/2} < 1$. If $|b|_p = p^{2k}$, then $\lambda = p^{1/2(p-1)}/p < 1$.

Set $S_\beta = \sum_{m=1}^{\infty} K_m \beta^m, \beta \in \mathcal{U}_{s(b)}$. As $\lambda < 1$, this operator belongs to the space $\mathcal{L}(L_2(Q_p, \nu_b))$ and moreover $\|S_\beta\| < 1$. As $T_\beta f = f + S_\beta f$ and $\|S_\beta f\| < \|f\|$, we get $\|T_\beta f\| = \max(\|f\|, \|S_\beta f\|) = \|f\|$. Hence the operator T_β is an isometry of the space $L_2(Q_p, \nu_b)$ for every $\beta \in \mathcal{U}_{s(b)}$.

5. Gaussian representations for the operators associated with position and momentum

As for ordinary Schrödinger quantum mechanics, let us define the coordinate and momentum operators in $L_2(Q_p, \nu_b)$ by

$$\hat{\boldsymbol{x}} f(x) = x f(x)$$

$$\hat{\boldsymbol{p}} f(x) = (-\mathrm{i}) \left(\frac{\mathrm{d}}{\mathrm{d}x} - (x/2b) \right) f(x)$$

where f belongs to the $Q_p(\mathrm{i})$-linear space $\mathcal{D}$ of linear combinations of Hermite polynomials.

The coordinate and momentum operators so defined satisfy $\mathcal{D}$ 'canonical commutation relations':

$$[\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}] = \mathrm{i}I \tag{7}$$

where I is the unit operator in $L_2(Q_p, \nu_b)$. We shall see that these relations can be extended on the whole of $L_2(Q_p, \nu_b)$.

Theorem 5.1. The operators of the coordinate $\hat{\boldsymbol{x}}$ and momentum $\hat{\boldsymbol{p}}$ are bounded in the space $L_2(Q_p, \nu_b)$ and

$$\|\hat{\boldsymbol{x}}\| = \sqrt{|b|_p} \qquad \|\hat{\boldsymbol{p}}\| = 1/\sqrt{|b|_p}. \tag{8}$$

Moreover $\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}$ are symmetric and satisfy (7) on $L_2(Q_p, \nu_b)$.

Proof. Let

$$f(x) = \sum_{n=0}^{\infty} f_n H_{n,b}(x) \in L_2(Q_p, \nu_b).$$

Then using the recurrence formula

$$H_{n+1,b}(x) = [x H_{n,b}(x) - n H_{n-1,b}(x)]/b \tag{9}$$

we get

$$\hat{\boldsymbol{x}} H_{n,b}(x) = b H_{n+1,b}(x) + n H_{n-1,b}(x). \tag{10}$$

and

$$\hat{\boldsymbol{x}} f(x) = \sum_{n=0}^{\infty} b f_n H_{n+1,b}(x) + \sum_{n=1}^{\infty} n f_n H_{n-1,b}(x).$$

Thus, using the strong triangle inequality we get

$$\begin{aligned} \|\hat{\boldsymbol{x}} f\|^2 &\leqslant \max\Big[\max_n |b|_p^2 |f_n|_p^2 |(n+1)!|_p / |b|_p^{n+1}, \max_n |n|_p^2 |f_n|_p^2 |(n-1)!|_p / |b|_p^{n-1}\Big] \\ &= |b|_p \max\Big[\max_n |n+1|_p |f_n|_p^2 |n!|_p / |b|_p^n, \max_n |n|_p |f_n|_p^2 |n!|_p / |b|_p^n\Big] \\ &\leqslant |b|_p \|f\|^2 \end{aligned}$$

(as $|n|_p \leqslant 1$ for all $n \in \mathbb{N}$). Thus $\|\hat{\boldsymbol{x}}\| \leqslant \sqrt{|b|_p}$. Now we prove that $\|\hat{\boldsymbol{x}}\|^2 = |b|_p$.

Let $n = p^k$, then

$$\sigma_{k,b} = \|\hat{\boldsymbol{x}} H_{p^k,b}\|^2 = \max(|b|_p^2 |(p^k+1)!|_p / |b|_p^{p^k+1}, |p^k|_p^2 |(p^k-1)!|_p | / |b|_p^{p^k-1}).$$

But $|(p^k+1)!|_p = |p^k!|_p$ and $|p^{2k}(p^k-1)!|_p = p^{-k}|p^k!|_p$. Thus $\sigma_{k,b} = |b|_p (|p^k!|_p / |b|_p^{p^k}) = |b|_p \|H_{p^k,b}\|^2$, which proves the first equality in (8).

Using equations (4) and (10), we get

$$\frac{\mathrm{d}}{\mathrm{d}x} H_{n,b}(x) = (x/b) H_{n,b}(x) - H_{n+1,b}(x) = (n/b) H_{n-1,b}(x)\,.$$

Set $\hat{T}_x = (\mathrm{d}/\mathrm{d}x - (x/2b))$. We have $\hat{T}_x H_{n,b}(x) = (n/2b) H_{n-1,b}(x) - (1/2) H_{n+1,b}(x)$. To compare this expression with (10), we rewrite it as

$$\hat{T}_x H_{n,b}(x) = (1/2b)[-b H_{n+1,b}(x) + n H_{n-1,b}(x)]\,. \tag{11}$$

The expression in square brackets is the same as in (10), the signum cannot play any role in the estimates of the max-type. That is why we can get $\|\hat{T}_x\| = (1/|b|_p)\|\hat{\boldsymbol{x}}\|$, which proves the second equality in (8).

The symmetry of $\hat{\boldsymbol{x}}, \hat{\boldsymbol{p}}$ is easily verified from their definition.

6. Unitary isometric one parameter operator groups corresponding to operators representing position and momentum

We shall compute $[\gamma(\hat{\boldsymbol{x}})]_p^-$ and $[\gamma(\hat{\boldsymbol{p}})]_p^-$.

If $|b|_p = p^{2k+1}$ then $\gamma(\hat{\boldsymbol{x}}) = 1/(p^k p^{1/2} p^{1/(p-1)})$. If $p \neq 3$ then $[\gamma(\hat{\boldsymbol{x}})]_p^- = 1/p^{k+1}$. If $p = 3$ then $[\gamma(\hat{\boldsymbol{x}})]_p^- = 1/p^{k+2}$. If $|b|_p = p^{2k}$ then $\gamma(\hat{\boldsymbol{x}}) = 1/(p^k p^{1/(1-p)})$ and $[\gamma(\hat{\boldsymbol{x}})]_p^- = 1/p^{k+1}$. Set $R(b) = [\gamma(\hat{\boldsymbol{x}})]_p^-$.

If $|b|_p = p^{2k+1}$ then $\gamma(\hat{\boldsymbol{p}}) = (p^{1/2}/p^{1/(p-1)}) p^k$. If $p \neq 3$ then $[\gamma(\hat{\boldsymbol{p}})]_p^- = p^k$. If $p = 3$ then $[\gamma(\hat{\boldsymbol{p}})]_p^- = p^{k-1}$. If $|b|_p = p^{2k}$ then $[\gamma(\hat{\boldsymbol{p}})]_p^- = p^{k-1}$.

Set $r(b) = [\gamma(\hat{\boldsymbol{p}})]_p^-$.

On the basis of propositions 2.1 and 2.2 and theorem 5.1 we easily get the following.

Theorem 6.1. The maps $\alpha \to U(\alpha) = \mathrm{e}^{\mathrm{i}\alpha\hat{\boldsymbol{x}}}, \alpha \in \mathcal{U}_{R(b)}$, and $\beta \to V(\beta) = \mathrm{e}^{\mathrm{i}\beta\hat{\boldsymbol{p}}}, \beta \in \mathcal{U}_{r(b)}$, are analytic one parameter groups of unitary isometric operators acting on $L_2(Q_p, \nu_b)$. They satisfy the Weyl commutation relations

$$U(\alpha)V(\beta) = \mathrm{e}^{-\mathrm{i}\alpha\beta} V(\beta)U(\alpha)\,. \tag{12}$$

Remark. The restrictions on the domains of the parameters α and β arise from the commutation factor $\mathrm{e}^{\mathrm{i}\alpha\beta}$. Furthermore we have

$$R(b)r(b) = \xi(1/p) \tag{13}$$

where $\xi(1/p) = 1/p^2$, if $|b|_p = p^{2k}$, and $\xi(1/p) = 1/p, p \neq 3$, if $|b|_p = p^{2k+1}$, and $\xi(1/3) = 1/27$ in the latter case. Thus the dependence on the covariance b of the Gaussian distribution has really disappeared. We can consider the relation (13) as a p-adic analogue of the Heisenberg uncertainty relations. It implies, in particular, that when $r(b) \to 0$ then automatically $R(b) \to \infty$ and vice versa.

Let us set

$$M_\beta f(x) = \mathrm{e}^{-\beta\hat{\boldsymbol{x}}/2b} f(x) = \sum_{n=0}^{\infty} \frac{(-\beta\hat{\boldsymbol{x}})^n}{n!(2b)^n} f(x) \tag{14}$$

for $f \in L_2(Q_p, \nu_b)$. Using proposition 2.1 and theorem 5.1, we easily get the following.

Proposition 6.1. The map $M : \mathcal{U}_{r(b)} \to IS(L_2(Q_p, \nu_b)), \beta \to M_\beta$, is an analytic one parameter group (indexed by the ball $\mathcal{U}_{r(b)}$).

Remark. Of course, the function $x \to \mathrm{e}^{-\beta x/2b}$ is not defined on whole Q_p and we cannot consider the operator (14) as an operator of pointweis multiplication.

7. Operator calculus

It is well known that in the ordinary $L_2(R, \mathrm{d}x)$ space the unitary group $V(\beta) = \mathrm{e}^{\mathrm{i}\beta\hat{p}}$, $\beta \in R$, can be realized as the translation group: $V(\beta)\psi(x) = \psi(x+\beta)$ for sufficiently good functions $\psi(x)$. If we consider the equivalent representation in L_2-space with respect to the Gaussian measure $\nu_b(\mathrm{d}x) = (\mathrm{e}^{-x^2/2b}/\sqrt{2\pi b})\mathrm{d}x$ on R, we get

$$V(\beta)\psi(x) = \mathrm{e}^{-\beta^2/4b}\mathrm{e}^{-\beta x/2b}\psi(x+\beta) \tag{15}$$

or

$$V(\beta) = c_\beta M_\beta T_\beta \tag{16}$$

where $c_\beta = \mathrm{e}^{-\beta^2/4b}$. We shall now prove that formula (16) is also valid in the p-adic case.

Set $S(\beta) = c_\beta M_\beta T_\beta$, $\beta \in \mathcal{U}_{r(b)}$, where the operator M_β is defined by (14).

Theorem 7.1. The map $\beta \to S_\beta$, $\beta \in \mathcal{U}_{r(b)}$,is one parameter analytic group of isometric unitary operators acting in $L_2(Q_p, \nu_b)$.

Proof. First the constant c_β defines an isometric multiplication operator in the space $L_2(Q_p, \nu_b)$ for every $\beta \in \mathcal{U}_{r(b)}$ because $|c_\beta|_p = 1$. On the basis of theorem 5.1 and proposition 6.1 we get that $S(\beta)$ is an isometric one parameter analytic group (because $r(b) \leqslant s(b)$). To prove the unitarity of this group, it is sufficient to show that $(S(\beta)x^n, S(\beta)x^n) = (x^n, x^n)$ for all monomials $x^n, n = 0, 1, \ldots$. By equation (14) we get

$$\begin{aligned}(S(\beta)x^n, S(\beta)x^n) &= \mathrm{e}^{-\beta^2/2b}(\mathrm{e}^{-\beta\hat{x}/2b}(x+\beta)^n, \mathrm{e}^{-\beta\hat{x}/2b}(x+\beta)^n)\\ &= \mathrm{e}^{-\beta^2/2b}\sum_{k,j=0}^{\infty}(-\beta/2b)^{k+j}(1/k!j!)\Gamma_{kj}(n)\end{aligned}$$

where $\Gamma_{kj}(n) = \int_{Q_p} x^{k+j}(x+\beta)^{2n}\nu_b(\mathrm{d}x)$. By change of variables, according to $y = x+\beta$, in the p-adic Gaussian integral [6] we get

$$\Gamma_{kj}(n) = \mathrm{e}^{-\beta^2/2b}\sum_{m=0}^{\infty}(\beta/b)^m(1/m!)\int_{Q_p}(y-\beta)^{k+j}y^{2n+m}\nu_b(\mathrm{d}y).$$

By proposition 6.1 we get

$$\begin{aligned}&(S(\beta)x^n, S(\beta)x^n)\\ &\quad = [\mathrm{e}^{-\beta^2/b}\sum_{m,k,j=0}^{\infty}(-\beta/2b)^{k+j}(\beta/b)^m(1/m!k!j!)(\hat{\boldsymbol{x}}^m(\hat{\boldsymbol{x}}-\beta)^k x^n, (\hat{\boldsymbol{x}}-\beta)^j x^n)\\ &\quad = \mathrm{e}^{-\beta^2/b}(\mathrm{e}^{\beta\hat{x}/b}\mathrm{e}^{-\beta(\hat{x}-\beta)/2b}x^n, \mathrm{e}^{-\beta(\hat{x}-\beta)/2b}x^n) = (x^n, x^n)\,.\end{aligned}$$

Lemma 7.1. The groups $S(\beta)$ and $V(\beta)$ have $\hat{\boldsymbol{p}}$ as their common generator.

Proof. We have $S'(\beta)|_{\beta=0} = -x/\beta + T'_\beta|_{\beta=0}$ with $T'_\beta|_{\beta=0} = \mathrm{d}/\mathrm{d}x$, so that $S'(\beta)|_{\beta=0} = \mathrm{d}/\mathrm{d}x - x/b$. Since both T'_β and $S'(\beta)$ are bounded this implies the result.

As a consequence of this lemma and the analyticity of the one parameter groups $S(\beta)$ and $V(\beta)$, we easily have the following.

Theorem 7.2. The representation (15), (16) holds for the operator group $V(\beta)$.

Equation (15) could serve as starting point for an operator quantization based on a calculus of pseudo-differential operators. Let us consider the expression

$$a(y,\xi)=\int_{\mathcal{U}_{\theta_1}}\int_{\mathcal{U}_{\theta_2}} d\mu(\alpha)\ d\mu(\beta)\ \tilde{a}(\alpha,\beta)\mathrm{e}^{\mathrm{i}y\alpha+\mathrm{i}\xi\beta} \tag{17}$$

defined first for smooth functions $\tilde{a}(\alpha,\beta)$ with compact support, $supp\ \tilde{a}\subset\mathcal{U}_{\theta_1}\times\mathcal{U}_{\theta_2}$. We denote by symbol $d\mu(x)$, where x is the integration variable, a Haar measure on Q_p with values in Q_p, see [17–19]. For every ball $\mathcal{U}_r$ this is a translation invariant additive set function defined on the algebra $\mathcal{F}(\mathcal{U}_r)$ generated by balls contained in $\mathcal{U}_r$. However, this measure is not bounded: $\sup\{|\mu(A)|_p : A\in\mathcal{F}(\mathcal{U}_r)\}=\infty$. Therefore, there exist continuos functions $f:\mathcal{U}_r\to Q_p$ which are not integrable with respect to this measure. However, the integral with respect to μ is well defined for C^1-functions $f:\mathcal{U}_r\to Q_p$. The integral representation (17) gives us the interesting connection between the domains of $\tilde{a}(\alpha,\beta)$ and $a(y,\xi)$.

Theorem 7.3. Let $\tilde{a}$ be the C^1 function and let $supp\ \tilde{a}\subset\mathcal{U}_{\theta_1}\times\mathcal{U}_{\theta_2}$. Then the symbol a defined by (17) is the C^1-function on the set $\mathcal{U}_{\bar{\theta}_1}\times\mathcal{U}_{\bar{\theta}_2}$ with $\bar{\theta}_j\theta_j\leqslant 1/p$, $j=1,2$.

The symbol $a(y,\xi)$ is not defined outside of the set $\mathcal{U}_{\bar{\theta}_1}\times\mathcal{U}_{\bar{\theta}_2}$. The physical interpretation of Theorem 7.3 is evident. This is the consequence of the uncertainty relation 'position–momentum'.

In analogy with the usual Weyl procedure (see, for example, [30]) we set

$$\begin{aligned}\hat{a}\phi(x)&=\int_{\mathcal{U}_{R(b)}}\int_{\mathcal{U}_{r(b)}} d\mu(\alpha)\ d\mu(\beta)\ \tilde{a}(\alpha,\beta)U(\alpha)V(\beta)\phi(x)\\ &=\int_{\mathcal{U}_{R(b)}}\int_{\mathcal{U}_{r(b)}} d\mu(\alpha)\ d\mu(\beta)\ \tilde{a}(\alpha,\beta)\exp\{\mathrm{i}\alpha x-\beta^2/4b-\beta x/2b\}\phi(x+\beta)\,.\end{aligned}$$

This formula realizes an analogue of the usual $\hat{\boldsymbol{x}}\hat{\boldsymbol{p}}$-quantization. In the same way we can realize the $\hat{\boldsymbol{p}}\hat{\boldsymbol{x}}$-quantization:

$$\begin{aligned}\hat{a}\phi(x)&=\int_{\mathcal{U}_{R(b)}}\int_{\mathcal{U}_{r(b)}} d\mu(\alpha)d\mu(\beta)\tilde{a}(\alpha,\beta)V(\beta)U(\alpha)\phi(x)\\ &=\int_{\mathcal{U}_{R(b)}}\int_{\mathcal{U}_{r(b)}} d\mu(\alpha)\ d\mu(\beta)\ \tilde{a}(\alpha,\beta)\exp\{\mathrm{i}\alpha(x-\beta)-\beta^2/4b-\beta x/2b\}\phi(x+\beta)\,.\end{aligned}$$

The problem of extending these relations to more general ('distributional') $\tilde{a}$ (in order to get, in particular, the polynomial symbols a) will be considered in further work.

8. On exactness of the measurement of positions and momenta

Let us consider the consequences of Theorem 5.1 concerning the measurement process. To simplify our considerations, we choose $|b|_p=p^{2k}, k=0,1,2,\ldots$. Then $\|\hat{\boldsymbol{x}}\|=p^k$ and $\|\hat{\boldsymbol{p}}\|=p^{-k}$.

If $|\lambda|_p>\|\hat{\boldsymbol{x}}\|=p^k$, then the resolvent operator $\hat{R}_\lambda=(\hat{\boldsymbol{x}}-\lambda I)^{-1}$ exists and λ does not belong to the spectrum of the coordinate operator. Thus, it would be impossible to mesure coordinates $\boldsymbol{x}=\lambda$ for such values of λ. By a similar reasoning we get, that it would be impossible to measure momenta $\boldsymbol{p}=\mu$ for μ such that $|\mu|_p>|h|_p p^{-k}$. But the condition $|\lambda|_p>p^k$ is equivalent to the canonical expansion

$$\lambda=\lambda_{-l}p^{-l}+\cdots+\lambda_0+\cdots\ \lambda_m p^m+\cdots \tag{18}$$

where $\lambda_j = 0, 1, \ldots, p-1, l = k+1, k+2, \ldots$ and $\lambda_{-l} \neq 0$. Let us remark that infinite p-adic fractions should be interpreted only as mathematical idealizations (in a similar way as for infinite real fractions), whereas physical values are in reality given by rational numbers of the type

$$\lambda = \lambda_{-l} p^{-l} + \cdots + \lambda_m p^m . \tag{19}$$

For the moment, let us consider a system of units where the Planck constant $h = 1$. Then we cannot measure momenta

$$\mu = \mu_l p^l + \cdots + \mu_m p^m \cdots \tag{20}$$

where $l = k-1, k-2, \ldots$ and $\mu_l \neq 0$ in the present Gaussian representation.

Computing the spectra of the p-adic position and momentum operators directly remains an open problem (we have only found balls which contain these spectra).

What are the possible physical interpretations of our results?

Our main result can be interpreted as stating that a non-Archimedean structure of space is equivalent to the finite exactness of a measurement.

This finite exactness may have different physical interpretations. First, we can consider our p-adic Hilbert space representation of quantum mechanics as a mathematical realization of the old idea in quantum mechanics that the result of a measurement involves also the influence of the measurement equipment. According to Bohm [31] 'the so-called 'observables' are not properties belonging to the observed system alone, but instead potentialities whose precise development depends just as much on the observing apparatus as on the observed system.' How can we include these properties of the observing apparatus into the structure of the space of quantum states? The p-adic Hilbert space representation is one of the possible ways. Here we may fix the exactness of the position (or momentum) measurement generated by the apparatus and construct the p-adic Hilbert space representation on the basis of this exactness.

Another physical interpretation of our main result is connected with the old idea of the existence of a 'fundamental length.' According to this point of view, the space is not homogeneous in the direction of the microworld and there exists some limiting value of lengths l_{fund} (such that any length $l < l_{\text{fund}}$ is deprived of any meaning). How can we describe such a situation in mathematical terms? There are several possibilities, see [32, 33], and the p-adic numbers are one of them. We have constructed (on the quantum level) the position representation where there exists a limiting measurable value of the position. One of the purposes of the 'fundamental length' hypotesis was to avoid all divergences in quantum field theory. Our p-adic approach (and related ones, see, e.g., [5–13, 22, 23, 28, 29]) operates in the same direction: all unbounded quantum mechanical operators (in the usual quantum mechanical formalism based on real and complex numbers) become bounded in the p-adic representation. The picture would be reproduced in a quantum field formalism built on the basis of p-adic numbers. Such a formalism will be described in our further investigations.

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